Stability of parallel flow in a parallel magnetic field at small magnetic Reynolds numbers

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The hydromagnetic stability of a basic two-dimensional parallel flow of an incompressible conducting fluid in a uniform magnetic field parallel to the flow is considered. By use of the generalization of the Orr–Sommerfeld equation for an electrically conducting fluid, it is shown that any given small wave disturbance can be stabilized by a sufficiently strong magnetic field if the Reynolds number is finite and the magnetic Reynolds number small.

Stability of velocity profiles with a point of inflexion at small magnetic Reynolds number and infinite Reynolds number is considered in detail. Perturbation methods are developed to find stability characteristics in two cases, when the magnetic field is weak, and when the disturbance is a long wave. These methods are applied to the jet and the half-jet, which are both found to be unstable to long-wave disturbances, however strong the magnetic field. Nonetheless, these two flows can be stabilized for any given harmonic disturbance of finite wavelength. The analysis of the jet reveals the surprising result that the magnetic field makes inviscid long-wave disturbances more *unstable*.

1. Introduction

Michael (1953) and Stuart (1954) were the first to consider the stability of a steady two-dimensional parallel flow of a viscous incompressible conducting fluid in a uniform magnetic field parallel to the flow. They supposed that the flow was bounded by perfectly conducting walls at $y = y_1, y_2$ (where y_1 and/or y_2 may be infinite if the flow is unbounded) and took a basic parallel flow with variable velocity

$$\mathbf{U} = (U(y), 0, 0) \quad (y_1 \leqslant y \leqslant y_2)$$

and a uniform parallel magnetic field

$$\mathbf{H}_0 = (H_0, 0, 0)$$

in cartesian components. Following the usual methods of hydrodynamic stability, they substituted the velocity and magnetic field of the basic flow together with a small disturbance (denoted by primes), say

$$\mathbf{u} = (u, v, w) = \mathbf{U} + \mathbf{u}'(\mathbf{x}, t) \tag{1.1}$$

and

$$\mathbf{H} = (H_x, H_y, H_z) = \mathbf{H}_0 + \mathbf{h}'(\mathbf{x}, t), \tag{1.2}$$

into the hydromagnetic equations for a homogeneous incompressible fluid of density ρ, kinematic viscosity ν, magnetic permeability μ, electrical conductivity

σ and magnetic diffusivity $\lambda = 1/4\pi\mu\sigma$. Then they linearized the equations by neglecting products of the primed quantities.

Squire's theorem that two-dimensional small disturbances of a parallel flow are the least stable is valid for a conducting fluid (Michael 1953; Stuart 1954). Therefore, in a proposed search for a sufficient condition for stability, only two-dimensional disturbances need be considered. Thus w, h'_z and $\partial/\partial z$ can be put equal to zero. Now the continuity equation $\nabla.\mathbf{u}' = 0$ and the Maxwell equation $\nabla.\mathbf{h}' = 0$ of the disturbance imply that there exist functions ψ', χ' such that

$$u' = -\partial\psi'/\partial y, \quad v' = \partial\psi'/\partial x, \tag{1.3}$$

and

$$h'_x = -\partial\chi'/\partial y, \quad h'_y = \partial\chi'/\partial x. \tag{1.4}$$

Assume that small disturbances can be resolved into dynamically independent wave-components by putting

$$\psi' = \phi(y)\exp\{i\alpha(x-ct)\}, \quad \chi' = \theta(y)\exp\{i\alpha(x-ct)\} \tag{1.5}$$

for some complex velocity $c = c_r + ic_i$ and positive wave-number α. This gives a wave with phase velocity c_r and logarithmic growth rate αc_i; so the motion is stable, neutrally stable or unstable according as c_i is respectively less than, equal to, or greater than zero.

If the velocity distribution $U(y)$ has velocity scale V and its space variation has length scale L, we can define dimensionless parameters for the flow, the first being the Reynolds number

$$R \equiv VL/\nu. \tag{1.6}$$

Analogously, we define the magnetic Reynolds number

$$R_M \equiv VL/\lambda. \tag{1.7}$$

This measures the ratio of terms representing convection and diffusion of the magnetic lines of force in the fluid. The ratio of magnetic to kinetic energy is measured by

$$S \equiv (\mu H_0^2/8\pi)/\tfrac{1}{2}\rho V^2; \tag{1.8}$$

S is also the square of the Alfvén velocity divided by V^2. It is useful to define a further dimensionless parameter

$$N \equiv SR_M = \mu H_0^2 L/4\pi\rho\lambda V. \tag{1.9}$$

By elimination of the hydrodynamic pressure, and by division of the appropriate dimensions out of the quantities y, U, c, α, ϕ and θ, it can be shown that the linearized hydromagnetic equations lead to

$$(U-c)\theta - \phi = -\frac{i}{\alpha R_M}(\theta'' - \alpha^2\theta) \tag{1.10}$$

and

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi - S(\theta'' - \alpha^2\theta) = -\frac{i}{\alpha R}(\phi^{\mathrm{iv}} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{1.11}$$

where primes now denote differentiation with respect to y. These equations were derived independently by Michael (1953) and Stuart (1954). On elimination of ϕ it at once follows that

$$\frac{i}{\alpha R_M}(D^2-\alpha^2)^3\theta+(U-c)(D^2-\alpha^2)^2\theta+4U'(D^2-\alpha^2)\theta' +2U''(D^2-\alpha^2)\theta+4U''\theta''+4U'''\theta'+U^{\mathrm{iv}}\theta$$
$$= -\frac{R}{R_M}\{(U-c)(D^2-\alpha^2)-U''\}(D^2-\alpha^2)\theta +i\alpha R\{(U-c)^2-S\}(D^2-\alpha^2)\theta+2i\alpha RU'(U-c)\theta', \qquad (1.12)$$

where $$D \equiv d/dy.$$

The stability characteristics can be found by solving the sixth-order differential equation (1.12) subject to the six appropriate boundary conditions. These are

$$\alpha\phi = 0 = \phi' \quad (y = y_1, y_2), \qquad (1.13)$$

which makes the velocity vanish at fixed walls, and

$$\alpha\theta = 0 \quad (y = y_1, y_2), \qquad (1.14)$$

which makes the normal component of the magnetic field vanish at perfectly conducting walls.

For given values of α, R, R_M and S we can (in principle) find the eigenvalue c. If $c_i < 0$ we conclude that the disturbance of wave-number α is stable for the given R, R_M and S. In practice it is convenient to put $c_i = 0$ and look for the resultant relation between α, R, R_M and S.

From equations (1.10) and (1.11) Stuart (1954) deduced the power equation of the disturbance

$$\frac{D}{Dt}\int \tfrac{1}{2}(\mathbf{u}'^2+S\mathbf{h}'^2)\,d\mathbf{x} = \int(-u'v'+Sh'_xh'_y)\frac{dU}{dy}\,d\mathbf{x} -\frac{1}{R}\int(\nabla\times\mathbf{u}')^2d\mathbf{x}-\frac{S}{R_M}\int(\nabla\times\mathbf{h}')^2d\mathbf{x}. \qquad (1.15)$$

This equation is a generalization of the Lorentz power equation for a non-conducting fluid. The first integral is the sum of the kinetic and magnetic energies of the disturbance, the second the rate of energy transfer *from* the basic parallel flow, the third the rate of viscous dissipation and the fourth the rate of ohmic dissipation of magnetic energy.

To see typical orders of magnitude of the dimensionless parameters, consider the following approximate values measured in a recent laboratory experiment (Murgatroyd 1953) on channel flow of mercury:

$$\lambda = 8\times10^3\,\mathrm{cm^2/sec},\quad \nu = 10^{-3}\,\mathrm{cm^2/sec},\quad \rho = 14\,\mathrm{g/cm^3},$$
$$V = 10\,\mathrm{cm/sec},\quad L = 1\,\mathrm{cm},\quad H_0 = 10^3\,\mathrm{g}.$$

These give parameters

$$R \approx 10^4,\quad R_M \approx 10^{-3},\quad N \approx 10^{-1}.$$

The value $R_M \approx 10^{-3}$ suggests the approximation $R_M \ll 1$ to simplify the formidable eigenvalue problem we are facing. Accordingly, let us balance the magnetic diffusion terms on the right-hand side of equation (1.10) with the convection of the basic magnetic field by the disturbance, i.e. make the approximation

$$\theta'' - \alpha^2\theta = -i\alpha R_M \phi. \tag{1.16}$$

This is equivalent to taking the first term of a power series of the form

$$\theta = \sum_{n=1}^{\infty} R_M^n \theta_n(y_j \alpha, R, N, c),$$

which is a regular expansion because R_M is the coefficient of terms other than the one of highest order in the differential equation (1.12). We shall suppose that N remains finite and non-zero as $R_M \to 0$, which requires $H_0 \to \infty$. Then θ can be eliminated from equation (1.11) by use of equation (1.16) to yield

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi + i\alpha N\phi = -\frac{i}{\alpha R}(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi). \tag{1.17}$$

This equation was first derived by Stuart (1954). It reduces to the Orr–Sommerfeld equation in the absence of a magnetic field. The convection of the magnetic field of the disturbance by the basic flow being neglected, the magnetic and velocity fields of the disturbance have been separated, and it is only necessary to solve equation (1.17) subject to the boundary conditions (1.13) in order to find c. θ is of order R_M and need not be found from equation (1.16) after ϕ has been found. However, the basic magnetic field, being infinite as $R_M \to 0$, affects the velocity of the disturbance and requires the addition of the term $i\alpha N\phi$ in the Orr–Sommerfeld equation.

The assumption $R_M = \nu R/\lambda \ll 1$ also requires

$$R \ll \lambda/\nu \quad (= 6 \times 10^6 \quad \text{for} \quad Hg).$$

In spite of this limitation of the range of Reynolds number, we shall consider later the inviscid form of the Stuart equation (1.17), namely

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi + i\alpha N\phi = 0, \tag{1.18}$$

together with the inviscid boundary conditions

$$\alpha\phi = 0 \quad (y = y_1, y_2) \tag{1.19}$$

representing zero normal velocity at the walls. The justification is that the solution of the inviscid equation (1.18) will give an asymptotic solution to the Stuart equation (1.17) for large αR, as the Rayleigh equation gives one to the Orr–Sommerfeld equation. In many cases this asymptotic solution should not differ much from the viscous solution in the range of chief practical interest, where R is as large as 10^4. Our knowledge of equation (1.18) for $N = 0$ suggests that these cases occur when the velocity profile has a point of inflexion, i.e. when the inviscid flow as well as the viscous flow is unstable. Also Stuart (1954) adapted Lin's method combining the 'inviscid' and 'viscous' solutions of the Orr–Sommerfeld equation, so the study of inviscid hydromagnetic stability is at least an important preliminary to the viscous problem.

It is to be expected that the effect of the longitudinal primary magnetic field is generally stabilizing. This idea is made precise and it is shown (§ 2) that, for $R_M = 0$,

$$N^3 > 27R/\{256\,\alpha^2(\max.\, U')^4\}$$

is a sufficient condition for stability. After § 2 we continue to take $R_M = 0$, but take $R = \infty$ as well.

For velocity profiles with a point of inflexion there is a neutrally stable solution of the Rayleigh equation. It is of the form $\phi = \phi_s(y)$, $\alpha^2 = \alpha_s^2 \neq 0$, $c = U_s$, where U_s is the value of U at the point of inflexion. The perturbation for small N of this known solution for $N = 0$ is considered in § 3. A general equation of the tangent to the curve of neutral stability (i.e. the curve $c_i(\alpha^2, N) = 0$) at $(\alpha_s^2, 0)$ in the (α^2, N)-plane is found. The tangent is inclined downward, so the local effect of the magnetic field is stabilizing.

In § 4 an approach to the problem for small values of the wave-number is described.

The analysis of §§ 3, 4 gives a rough idea of the relation between α and N for each value of c_i, and for $c_i = 0$ in particular. This relation could be plotted as curves of constant c_i in the (α, N)-plane. We can find the values of c_i on the α-axis by solution of the Rayleigh equation. We can find the values near $(\alpha_s, 0)$ from § 3, and for small α from § 4. This may lead to our chief aim—to find if there is some critical value of N above which the flow is stable to disturbances of all wave-numbers. The solution for small α may show that this critical value does not exist, i.e. that for any value of N there exists some α for which the flow is unstable. If N has a critical value, it may be found by an explicit solution or by computation.

The ideas of §§ 3, 4 are applied in §§ 5, 6 to two important types of velocity profile with a point of inflexion and with $\nu = 0$. Helmholtz flow (unbounded uniform parallel flow with a single discontinuity of velocity) is found (§ 5) to be unstable for all values of the magnetic field at $R_M = 0$. Thus there is no critical value of N. However, the magnetic field makes the flow less unstable. Helmholtz flow is used to approximate the half-jet (with $U = \tanh y$, say), which is shown to be unstable similarly.

Also a broken-line velocity profile is used (§ 6) to approximate the jet (with $U = \operatorname{sech}^2 y$). This flow is unstable however large the magnetic field. For small wave-numbers it is found that the magnetic field *increases* the instability. This surprising result is discussed after the general conclusions of § 7.

2. Sufficient conditions for stability

Some fundamental stability characteristics of the viscous flow can be found by generalizing Synge's (1938) sufficient conditions for stability of parallel flow. We adapt this method for the Orr–Sommerfeld equation (see also Lessen 1952) to the Stuart equation. Thus the power equation (1.15) for $R_M = 0$ can be written, after use of (1.16), as

$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = -\tfrac{1}{2} i\alpha R(Q - Q^*) - \alpha R c_i (I_1^2 + \alpha^2 I_0^2) - \alpha^2 R N I_0^2, \qquad (2.1)$$

where
$$I^2 \equiv \int_{y_1}^{y_2} |d^n\phi/dy^n|^2\, dy \quad (n = 0, 1, 2), \qquad (2.2)$$

$$Q \equiv \int_{y_1}^{y_2} \{U\,|\phi'|^2 + (\alpha^2 U + U'')\,|\phi|^2\}\, dy + \int_{y_1}^{y_2} U'\phi'\phi^*\, dy, \qquad (2.3)$$

and an asterisk denotes a complex conjugate.

For inviscid fluid ($R = \infty$), this becomes

$$c_i(I_1^2 + \alpha^2 I_0^2) = \int_{y_1}^{y_2} \tfrac{1}{2} i U'(\phi'\phi^* - \phi'^*\phi)\,dy - \alpha N I_0^2. \tag{2.4}$$

Therefore the flow is always stable if

$$\alpha N \geqslant \max.\left(\int_{y_1}^{y_2} \tfrac{1}{2} i U'(\phi'\phi^* - \phi'^*\phi)\,dy \Big/ \int_{y_1}^{y_2} |\phi|^2\,dy\right),$$

the maximum being with respect to any class of functions including the eigenfunctions of the inviscid equation (1.18). Unfortunately equation (1.18) has a singularity where $U = c$ and is not complex self-adjoint, and so does not appear to have a variational principle suitable for computation of eigenvalues.

Returning to Synge's method, we can deduce from equation (2.1) that a sufficient condition for stability ($c_i < 0$) is

$$q\alpha R < \min.\{(I_2^2 + 2\alpha^2 I_1^2 + \alpha^2(\alpha^2 + RN)\,I_0^2)/I_0 I_1\}, \tag{2.5}$$

where

$$q \equiv \max.\, U', \tag{2.6}$$

because

$$\tfrac{1}{2}\,|Q - Q^*| \leqslant q I_0 I_1$$

by Schwarz's inequality. To exploit inequality (2.5) further, Synge noted that

$$\int_{y_1}^{y_2} \left(\phi + \frac{2\tilde{\xi}}{y_2 - y_1} y\phi' + \eta\phi''\right)\left(\phi^* + \frac{2\tilde{\xi}}{y_2 - y_1} y\phi'^* + \eta\phi''^*\right) dy > 0 \tag{2.7}$$

for all real $\tilde{\xi}$, η. We must add the proviso that $y_2 - y_1 < \infty$. If one or both of the boundaries are at infinity, we can use the modifications of Lessen (1952), whose strongest result comes with $\tilde{\xi} = 0$. The cases of bounded and unbounded flow can be combined by putting $\xi = 2\tilde{\xi}/(y_2 - y_1)$, which must be zero for unbounded flow. Therefore, on integration of the inequality (2.7) by parts and on further use of Schwarz's inequality,

$$\eta^2 I_2^2 > (\xi\eta - \xi^2 + 2\eta)\,I_1^2 + (\xi - 1)\,I_0^2. \tag{2.8}$$

Combination of the inequalities (2.5) and (2.8) shows that the motion is stable provided

$$\eta^2\alpha R c_i(I_1^2 + \alpha^2 I_0^2) < \eta^2 q\alpha R I_0 I_1 - I_1^2(2\alpha^2\eta^2 + \xi\eta - \xi^2 + 2\eta) - I_0^2(\eta^2\alpha^4 + \alpha^2\eta^2 RN + \xi - 1). \tag{2.9}$$

Stability is assured if the right-hand side is negative definite, i.e. if

$$(\eta^2 q\alpha R)^2 < 4(2\alpha^2\eta^2 + \xi\eta - \xi^2 + 2\eta)\,(\alpha^4\eta^2 + \alpha^2 RN\eta^2 + \xi - 1) \tag{2.10}$$

for all real ξ, η satisfying

$$2\alpha^2\eta^2 + \xi\eta - \xi^2 + 2\eta > 0, \quad \alpha^4\eta^2 + \alpha^2 RN\eta^2 + \xi - 1 > 0. \tag{2.11}$$

Note that the inequality (2.10) is weakened if we put $N = 0$.

For bounded flows we can get

$$(qR)^2 < 8(\alpha^2 + 1)\,(\alpha^2 + RN) \tag{2.12}$$

when $\xi = 1 = \eta$, and

$$(qR)^2 < (2\alpha^2 + 1)\,(4\alpha^4 + 4\alpha^2 RN + 1)/\alpha^2 \tag{2.13}$$

when $\xi = 2 = \eta$. These two inequalities limit the region of possible instability in the (α, R)-plane for each value of N, inequality (2.12) showing that the flow is stable if $R < 8N/q^2$.

For bounded or unbounded flows we can put $\xi = 0, \eta = (3/RN)^{\frac{1}{2}}/\alpha$ in inequality (2.10) to get

$$N^3 > 27R/256\,\alpha^2 q^4. \tag{2.14}$$

Thus, given any finite R and $\alpha \neq 0$, we can find N such that the disturbance is stable.

3. Perturbation of the neutral solution of Rayleigh's equation

Rayleigh proved that, if the equation

$$\phi'' - \alpha^2\phi - \frac{U''}{U-c}\phi = 0 \tag{3.1}$$

has a neutral eigensolution (i.e. one with an eigenvalue $c_i = 0$), then

$$U''(y_s) = 0 \tag{3.2}$$

at some point y_s of the flow. Later Tollmien showed that when (3.2) is satisfied there exists an eigensolution

$$\phi = \phi_s, \quad c = U_s = U(y_s), \quad \alpha = \alpha_s > 0, \tag{3.3}$$

for certain functions $U(y)$, where ϕ_s is real. A perturbation of this neutral oscillation with non-zero wave-number (carried out first by Tollmien, and later by Lin) establishes the existence of amplified oscillations with slightly different wave-number. Lin (1955) has given the following formula, which we shall use:

$$(dc/d\alpha^2)_{\alpha=\alpha_s} = \left\{\int_{y_1}^{y_2} \phi_s^2\, dy\right\} \Big/ \left\{\mathfrak{P}\int_{y_1}^{y_2} \frac{K\phi_s^2}{U-U_s}\, dy + i\pi K(y_s)\,\phi_s^2(y_s)\right\}, \tag{3.4}$$

where $\mathfrak{P}$ denotes the principal value of the integral and

$$K \equiv -U''(y)/\{U(y)-U_s\}, \tag{3.5}$$

K being positive for the profiles under consideration. The sign of the imaginary term in (3.4) depends on taking the limit $c_i \to 0$ through positive values in order to get the correct inviscid limit of the viscous solution.

Our aim is to perturb the neutral solution (3.3) in order to find the stability when the magnetic field is weak. So use† the Taylor series

$$c = U_s + (\alpha^2 - \alpha_s^2)\frac{\partial c}{\partial \alpha^2} + \alpha N \frac{\partial c}{\partial(\alpha N)} + \frac{1}{2!}(\alpha^2-\alpha_s^2)^2 \frac{\partial^2 c}{\partial(\alpha^2)^2} + \ldots, \tag{3.6}$$

the derivatives all being evaluated at $\alpha = \alpha_s$, $N = 0$. Then a neutral disturbance with $\alpha \neq \alpha_s$, $N \neq 0$ must be such that

$$\alpha_s N \sim -(\alpha^2 - \alpha_s^2)\left(\frac{\partial c_i/\partial \alpha^2}{\partial c_i/\partial(\alpha N)}\right)_{\alpha=\alpha_s,\, N=0} \quad \text{as} \quad \alpha \to \alpha_s. \tag{3.7}$$

We shall now find $\partial c/\partial(\alpha N)$ and combine it with the relation (3.4) for $\partial c/\partial \alpha^2_{\alpha=\alpha_s,\, N=0}$.

† This method is analogous to that for large Reynolds number used by Lessen & Fox (1955).

Subtract the product of ϕ_s and the equation

$$\phi'' - \alpha_s^2\phi - \frac{U''}{U-c}\phi + \frac{i\alpha_s N}{U-c}\phi = 0 \tag{3.8}$$

from the product of ϕ and the equation

$$\phi_s'' - \alpha_s^2\phi_s - \frac{U''}{U-U_s}\phi_s = 0. \tag{3.9}$$

Then, on integration between the boundaries, we obtain

$$i\alpha_s N\int_{y_1}^{y_2}\frac{\phi\phi_s}{U-c}dy = (c-U_s)\int_{y_1}^{y_2}\frac{U''\phi\phi_s}{(U-c)(U-U_s)}dy. \tag{3.10}$$

On taking the limit $c \to U_s$ such that $c_i \to 0$ through positive values, it can be shown that

$$\frac{\partial c}{\partial(\alpha N)_{\alpha=\alpha_s,\, N=0}} = -i\left\{\mathfrak{P}\int_{y_1}^{y_2}\frac{\phi_s^2}{U-U_s}dy + i\pi\phi_s^2(y_s)\right\}$$
$$\div\left\{\mathfrak{P}\int_{y_1}^{y_2}\frac{K\phi_s^2}{U-U_s}dy + i\pi K(y_s)\,\phi_s^2(y_s)\right\}. \tag{3.11}$$

Combination of equations (3.4) and (3.11) with the limit (3.7) gives the tangent to the curve of neutral stability in the (α^2, N)-plane at the point $(\alpha_s^2, 0)$ as

$$N = \frac{\alpha_s^2-\alpha^2}{\alpha_s}\left\{\pi K(y_s)\,\phi_s^2(y_s)\int_{y_1}^{y_2}\phi_s^2\,dy\right\}$$
$$\div\left\{\mathfrak{P}\int_{y_1}^{y_2}\frac{\phi_s^2}{U-U_s}dy\,\mathfrak{P}\int_{y_1}^{y_2}\frac{K\phi_s^2}{U-U_s}dy + \pi^2K(y_s)\,\phi_s^4(y_s)\right\}. \tag{3.12}$$

In cases of interest we shall find that these integrals are positive or zero, so that α^2 decreases from α_s^2 as N increases from zero. A disturbance with wave-number $\alpha < \alpha_s$ is unstable when $N = 0$ and, it now appears, is neutral for some $N > 0$; thus the effect of a small magnetic field is stabilizing.

4. Solution for small wave-numbers

When α is small the disturbance is a long wave, and the disturbance velocity is nearly parallel to the basic flow because

$$v' = i\alpha\phi\exp\{i\alpha(x-ct)\} \to 0 \quad \text{as} \quad \alpha \to 0.$$

On the large scale of a long wave the detailed structure of the velocity profile is unimportant in determining the stability characteristics. This property in the absence of a magnetic field is illustrated by a comparison, made by Carrier (cf. Esch 1957), of the stability properties of various forms of the velocity profile of the half-jet. He showed that the three unbounded profiles

$$U = y/|y|,\ \begin{cases} y/|y| & (|y|>1), \\ y & (|y|<1), \end{cases}\ 2\,\mathrm{erf}\,y - 1 \tag{4.1}$$

have the same inviscid eigenvalues ($c \to \pm i$) as $\alpha \to 0$. In fact the velocity profiles

$$U = \begin{cases} 0 & (|y|>1), \\ 1 & (|y|<1), \end{cases}\ \begin{cases} 0 & (|y|>1), \\ 1-|y| & (|y|<1), \end{cases}\ \mathrm{sech}^2\,y \tag{4.2}$$

of a jet have similar stability characteristics ($c \sim \pm\alpha^{\frac{1}{2}} i,\ \pm(\alpha/3)^{\frac{1}{2}} i,\ \pm(2\alpha/3)^{\frac{1}{2}} i$, respectively) as $\alpha \to 0$.

The addition of the magnetic field does not affect any term involving a derivative of U in the inviscid equation (1.18). Therefore, in looking for the stability of the half-jet and jet to long-wave disturbances, we may use the simple first profiles of (4.1) and (4.2) rather than the complicated third profiles. Other flows can be approached in this way, as first demonstrated by Rayleigh (1945).

5. The half-jet

Consider the velocity profile

$$U = y/|y| \quad (\infty > y > -\infty) \tag{5.1}$$

as a limit of the smoothly varying profile of the half-jet. This case has been treated by Michael (1955) for $R_M = \infty$ and by Nisbet (1960) for a two-fluid model for all values of R_M.

There is a discontinuity of the solution at $y = 0$. As with ordinary Helmholtz instability (cf. Rayleigh 1945), the condition that a particle stays in the interface

$$y = Y_0 \exp\{i\alpha(x-ct)\}$$

is

$$Y_0 = -\{\phi/(U-c)\}_{y=0}$$

to first order of the small disturbance. Therefore $\phi/(U-c)$ is continuous at $y = 0$.

From equation (1.18)

$$[(U-c)\phi' - U'\phi]_{-\epsilon}^{\epsilon} = \int_{-\epsilon}^{\epsilon} \{\alpha^2(U-c) - i\alpha N\}\phi\, dy.$$

We are supposing that the smoothly varying profile tends to the broken-line profile (5.1), during which limit the integrand on the right-hand side is bounded. On letting $\epsilon \to 0$, it follows that $(U-c)\phi' - U'\phi$ is continuous at $y = 0$. This condition could be alternatively proved (cf. Rayleigh 1945) by the continuity of the hydrodynamic pressure across the interface.

For the velocity distribution (5.1)

$$(\pm 1 - c)(\phi'' - \alpha^2\phi) + i\alpha N\phi = 0 \quad (\pm y > 0). \tag{5.2}$$

Now

$$\alpha\phi \to 0 \quad \text{as} \quad y \to \pm\infty,$$

because the disturbance must die away at infinity. Therefore

$$Q = \begin{cases} A\exp\{-y[\alpha^2 - i\alpha N/(1-c)]^{\frac{1}{2}}\} & (y \to 0), \\ B\exp\{y[\alpha^2 + i\alpha N/(1+c)]^{\frac{1}{2}}\} & (y < 0), \end{cases} \tag{5.3}$$

where the square-roots are chosen to have positive real parts. The continuity of $\phi/(U-c)$ and $(U-c)\phi' - U'\phi$ at $y = 0$ implies that $(U-c)^2\phi'/\phi$ is continuous there; whence it can be seen from the solution (5.3) that the eigenvalue relation is

$$(1-c)^2[\alpha^2 - i\alpha N/(1-c)]^{\frac{1}{2}} + (1+c)^2[\alpha^2 + i\alpha N/(1+c)]^{\frac{1}{2}} = 0, \tag{5.4}$$

i.e.

$$f(c) \equiv -ic(1+c^2) + (N/4\alpha)(1+3c^2) = 0. \tag{5.5}$$

We can tabulate the key points of this cubic on the imaginary axis as follows

c/i	$-\infty$	-1	$-1/3^{\frac{1}{2}}$	0	$1/3^{\frac{1}{2}}$	1	$+\infty$
f	$+\infty$	$-N/2\alpha$	$-2/3^{\frac{3}{2}}$	$N/4\alpha$	$2/3^{\frac{3}{2}}$	$-N/2\alpha$	$-\infty$

It can be seen that all three roots c_1, c_2, c_3 are pure imaginary, with

$$-\infty < c_{i1} < -1, \quad -1/3^{\frac{1}{2}} < c_{i2} < 0, \quad 1/3^{\frac{1}{2}} < c_{i3} < 1.$$

If $N = 0$, $c_i = \pm 1$ or 0. The roots $c = \pm i$ are those found for Helmholtz instability with no magnetic field. The other root $c_2 = 0$ is inadmissible because it comes from implicitly using a square-root with negative real part after squaring equation (5.4).

If $\alpha/N \to 0$, then $c \to \pm i/3^{\frac{1}{2}}$ or $-3iN/4\alpha$. Discarding the root c_2, we get

$$c \sim i/3^{\frac{1}{2}} \quad \text{or} \quad -3iN/4\alpha. \tag{5.6}$$

In all cases the flow is unstable, because $c_{i3} > 0$. As N increases from zero to infinity, c_{i3} decreases from 1 to $3^{-\frac{1}{2}}$. Thus the magnetic field makes the flow less unstable.

We can take

$$U = \tanh y \quad (\infty > y > -\infty) \tag{5.7}$$

as a smoothly varying representation of the velocity profile of the half-jet. The neutral eigensolution of the Rayleigh equation is then (Curle 1956)

$$\phi_s = \operatorname{sech} y, \quad \alpha_s = 1, \quad c = 0. \tag{5.8}$$

Therefore $y_s = 0$, $\phi_s(y_s) = 1$, $K = 2\operatorname{sech}^2 y$ and the analysis of § 3 gives

$$N \sim 2(1-\alpha^2)/\pi \tag{5.9}$$

as $\alpha \to 1$ on the curve of neutral stability.

The result

$$c \to 3^{-\frac{1}{2}} i \quad \text{or} \quad -\tfrac{3}{4} iN/\alpha \quad \text{as} \quad \alpha \to 0 \tag{5.10}$$

for Helmholtz flow has been shown in § 4 to apply to the half-jet. This can be laboriously verified by use of the second velocity profile (4.1). In all cases (5.10) holds, showing that the half-jet is unstable, however strong the magnetic field.

6. The jet

The velocity of the broken-line jet

$$U = \begin{cases} 0 & (|y| > 1), \\ 1 & (|y| < 1), \end{cases} \tag{6.1}$$

is an even function of y. Therefore the even and odd parts of ϕ are separable in equation (1.18) and can satisfy the boundary conditions separately. We shall consider even ϕ (corresponding to an antisymmetric disturbance) only, because it can be shown (in the same way as for even ϕ) that odd ϕ give rise to weaker instability. Therefore, in looking for a sufficient condition for stability, we take the even solution of equation (1.18) which satisfies the boundary condition at infinity. This solution is defined by

$$\phi = \begin{cases} D \exp\{[\alpha^2 + i\alpha N/c]^{\frac{1}{2}}(1-y)\} & (y > 1), \\ E \cosh\{[\alpha^2 - i\alpha N/(1-c)]^{\frac{1}{2}} y\} & (1 > y \geqslant 0). \end{cases} \tag{6.2}$$

Also $(U-c)^2\phi'/\phi$ must be continuous at $y = 1$, by the interfacial boundary conditions. Therefore

$$-c^2[\alpha^2+i\alpha N/c]^{\frac{1}{2}} = (1-c)^2[\alpha^2-i\alpha N/(1-c)]^{\frac{1}{2}}\tanh[\alpha^2-i\alpha N/(1-c)]. \quad (6.3)$$

If $N = 0$, we get Rayleigh's solution with

$$c = 1/(1 \pm i\coth^{\frac{1}{2}}\alpha) \quad (6.4)$$

and therefore instability for all α. If we fix $N \neq 0$ and let $\alpha \to 0$, it can be shown that

$$c/(\alpha N)^{\frac{1}{3}} \to i^{\frac{1}{3}} = \tfrac{1}{2}(3^{\frac{1}{2}}+i), \quad \tfrac{1}{2}(-3^{\frac{1}{2}}+i) \quad \text{or} \quad -i.$$

The second root is inadmissible because it corresponds to the square-root on the left-hand side of equation (6.3) with negative real part, i.e. to an exponentially increasing disturbance as $y \to \infty$. Therefore

$$c \sim -i(\alpha N)^{\frac{1}{3}} \quad \text{or} \quad \tfrac{1}{2}(3^{\frac{1}{2}}+i)(\alpha N)^{\frac{1}{3}} \quad \text{as} \quad \alpha \to 0. \quad (6.5)$$

The first root is stable ($c_i < 0$) and must join up with the stable root $c = 1/(1+i[\coth\alpha]^{\frac{1}{2}})$ of equation (6.4). But the second root is unstable and becomes *more* unstable as N increases. Thus we have found an unstable solution for all values of N, and the flow is therefore unstable. The strange circumstance that the magnetic field increases instability is discussed in § 7.

The continuous velocity profile of the jet (as found by Bickley—see Savic 1941) is

$$U = \operatorname{sech}^2 y \quad (\infty > y > -\infty), \quad (6.6)$$

and the neutral even eigensolution of the Rayleigh equation is

$$\phi_s = \operatorname{sech}^2 y, \quad \alpha_s = 2, \quad c = \tfrac{2}{3}. \quad (6.7)$$

After a little integration and computation, it can be shown that for this case equation (3.12) gives

$$\begin{aligned} N &= 6\pi(4-\alpha^2)/\{2\pi^2+81[1+3^{-\frac{1}{2}}\ln(2+3^{\frac{1}{2}})][2+3^{-\frac{1}{2}}\ln(2+3^{\frac{1}{2}})]\} \\ &= 0{\cdot}045(4-\alpha^2) \end{aligned} \quad (6.8)$$

for the tangent to the curve of neutral stability at (4, 0) in the (α^2, N)-plane. The steepness of the gradient of the line indicates a strong stabilizing influence of the magnetic field.

For small α we expect qualitative agreement with the limit (6.5) of c for the broken-line jet. That limit can be confirmed by use of the second profile (4.2).

7. Conclusions

The results of §§ 5, 6 can be pieced together to give a good picture of the inviscid stability characteristics of the half-jet and jet. The solution of the Rayleigh equation gives c_i on the α-axis in the (α, N)-plane, and, in particular, it gives $c_i = 0$ at $(\alpha_s, 0)$. The analysis of § 3 has been used to give the curve of neutral stability near $(\alpha_s, 0)$. If the curve leaves this point and eventually passes through the origin or cuts the N-axis, there will be a critical value of N above which $c_i < 0$ for all α. In fact this is not so for the half-jet or jet, since we have shown in §§ 5, 6 that these flows are unstable for small α, however large N is. Now the sufficient condition (2.14) for stability implies that, for given finite R and non-zero α, the flow can be stabilized. Therefore, for given $\alpha \neq 0$, it can be stabilized at infinite Reynolds number by the argument of continuity. In this way it is deduced that the curves of neutral stability for the jet and half-jet have the form shown in figure 1, with no critical value of N.

Finite viscosity and magnetic diffusivity cannot be expected to stabilize long-wave disturbances, because the viscous and diffusion terms in the hydromagnetic equations involve second derivatives of the disturbance fields. However, there are alleviating circumstances that might give stability in practice. First, only long waves are unstable, and disturbances of great wavelength are not often excited by slight irregularities in an experiment. Secondly, the growth rate $\exp(\alpha c_i t)$ of the disturbance is small, because c_i is $O(\alpha^{\frac{1}{3}})$ for the jet and $O(1)$ for the half-jet as $\alpha \to 0$. Therefore unstable disturbances may be carried downstream before they have time to grow appreciably. These qualitative ideas indicate that jets may be stabilized in practice, the half-jet being more unstable than the jet.

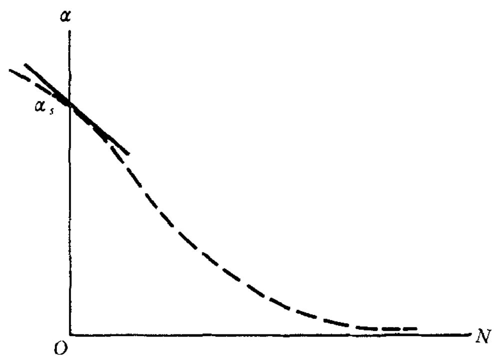

FIGURE 1. Conjectured form of the curve of neutral stability for the jet and half-jet.

It was found in § 6 that the jet was unstable, with $c_i \sim \frac{1}{2}(\alpha N)^{\frac{1}{3}}$ as $\alpha \to 0$, for $R_M = 0$, $R = \infty$ and finite $N \neq 0$. Therefore, if N is increased, c_i increases, i.e. the flow becomes more *unstable* to long-wave disturbances, contrary to our impression of the stabilizing tendency of magnetic fields. It should be remembered that this result is invalid if N is $O(\alpha)$ or $O(1/\alpha)$.

We should really be no more surprised at the creation of instability by the magnetic field than that by viscosity (the latter effect being known to occur under certain circumstances). In the complex dynamics of the flow, the magnetic field modifies the disturbance so that there is simultaneously magnetic energy dissipation and a different energy transfer from the basic flow to the disturbance by the Reynolds stress. Indeed, if the magnetic field were only a stabilizing agent, there should be an extremal principle for the energy as proof, contrary to the inconclusive power equation (2.4).

To verify this interpretation, we may use the power equation to give the rate of energy transfer from the basic flow per unit length in the z-direction as

$$P = \int_{-\infty}^{\infty} \tau \,(dU/dy)\, dy, \tag{7.1}$$

where the (dimensionless) Reynolds stress is

$$\tau \equiv -\overline{u'v'} \equiv -\frac{1}{2\pi}\int_0^{2\pi} u'v'\,dx$$
$$= \tfrac{1}{2}i\alpha\, e^{2\alpha c_i t}(\phi\phi'^* - \phi'\phi^*) \qquad (7.2)$$

(we discount the magnetic Reynolds stress $S\overline{h_x' h_y'}$ because $|\mathbf{h}'| = O(R_M \phi)$ when R_M is small).

For the broken-line jet (6.1),

$$dU/dy = \delta(-1) - \delta(1)$$

in terms of the Dirac δ-functions; also the disturbance found in § 6 is anti-symmetric. Therefore

$$P = -2(\tau)_{y=1}. \qquad (7.3)$$

Use of discontinuous profiles such as (6.1) leads to ambiguity (for flows with or without a magnetic field) because even though $\phi/(U-c)$ and $(U-c)\phi'$ are continuous, $(\phi\phi'^* - \phi'\phi^*)$ is discontinuous if $c_i \neq 0$. Thus use of discontinuous profiles cannot give precise information on the second-order quantity τ. However, we can find the trend for a smoothly varying profile of the jet $U = \operatorname{sech}^2 y$ by evaluating τ on both sides of the discontinuity of the known solution (6.2).

It can be shown from § 6 that

$$\alpha^{-1} e^{-2\alpha c_i t}(\tau)_{y\to 1} = -|D|^2 \operatorname{Im}\{[\alpha^2 + i\alpha N/c]^{\frac{1}{2}}\}$$

or

$$-|D|^2 \operatorname{Im}\left\{\frac{c(1-c^*)}{c^*(1-c)}[\alpha^2 + i\alpha N/c]^{\frac{1}{2}}\right\},$$

according as $y \to 1$ from above or below respectively, for the arbitrary constant D. If $\alpha \to 0$ for given $N \neq 0$, the unstable mode found in § 6 has $c \sim \frac{1}{2}(3^{\frac{1}{2}} + i)(\alpha N)^{\frac{1}{3}}$. Therefore

$$(\tau)_{y\to 1}/[(\alpha^4 N)^{\frac{1}{3}}\,|D|^2 \exp\{(\alpha^4 N)^{\frac{1}{3}} t\}] \to -\tfrac{1}{2} \quad \text{or} \quad -2.$$

On both sides τ is negative, so the tendency of the magnetic field is to increase the energy of the disturbance, as shown by equation (7.3).

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